MATH 821, Spring 2013, Lecture 7

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Recall from Lecture 6:

Proposition 1. *Suppose* $T(x) \in \mathbb{R}^{\geq 0}[[x]]$, $E(x,y) \in \mathbb{R}^{\geq 0}[[x,y]]$ *with*

- $E(0,0) = 0$,
- *E has a term of degree > 1 in y,*
- $\frac{d}{dx}E(x, T(x)) \neq 0$ *(so since coefficients are nonnegative, in particular* $\frac{d}{dx}E(\rho, T(\rho)) \neq 0$*)*

and $T(x) = E(x, T(x))$*, as formal power series. Let ρ be the radius of convergence of $T(x)$ and suppose $0 < \rho < \infty$, $T(\rho) < \infty$ and $\exists\, \epsilon$ such that $E(\rho + \epsilon, T(\rho) + \epsilon) < \infty$. Then $\exists$ functions $A(x), B(x)$ analytic at 0 such that*

$$T(x) = A(\rho - x) + B(\rho - x)\sqrt{\rho - x}$$

for $|x| < \rho$, x near ρ.

1 Proof of the square root result

Theorem 2 (Weierstraß preparation)**.** *Let* $f : \overset{x}{\mathbb{C}} \times \overset{y}{\mathbb{C}} \to \mathbb{C}$ *and let f be analytic in a neighbourhood of* $(0,0)$*. Suppose*

$$f(0,0) = \frac{d}{dy}f(0,0) = \cdots = \frac{d^{k-1}}{dy^{k-1}}f(0,0) = 0, \text{ but } \frac{d^k}{dy^k}f(0,0) \neq 0.$$

Then in a neighbourhood of $(0,0)$ *we can uniquely write* $f(x,y) = p(x,y)r(x,y)$ *where*

- *p, r analytic in the neighbourhood*
- *r is nowhere 0 in the neighbourhood*
- $p(x,y) = p_0(x) + p_1(x)y + \cdots + p_{k-1}y^{k-1} + y^k$ *(a Weierstraß polynomial) with the p_i analytic in a neighbourhood of 0 and* $p_i(0) = 0$

Sketch of proof. (For details see analysis text.)
Unique by expanding series.
By conditions on f,

- $\frac{d^k}{dy^k} f(x, y)$ is nonzero at $(0, 0)$, so there exists a small neighbourhood of $(0, 0)$ where it is nowhere 0,
- $f(0, y)$ has a root at 0 of multiplicity k, so for fixed x_0 sufficiently near 0, $f(x_0, y)$ has k roots (maybe distinct)

So there exists a Weierstraß polynomial with the same root structure, call it $p(x, y)$. Then

$$\frac{f(x, y)}{p(x, y)}$$

is analytic and nowhere 0 in a neighbourhood of $(0, 0)$. □

Corollary 3 ($k = 1$ in Weierstraß preparation, Implicit function theorem)**.**
Let $f : \overset{x}{\mathbb{C}} \times \overset{y}{\mathbb{C}} \to \mathbb{C}$ *and let* f *be analytic in a neighbourhood of* $(0, 0)$*. Suppose*

$$f(0, 0) = 0, \;\; \textit{but} \; \frac{d}{dy} f(0, 0) \neq 0.$$

Then there exists a neighbourhood of 0 in $\mathbb{C}$ *and a function* $g(x)$ *analytic on the neighbourhood with*

(1) $f(x, g(x)) = 0$*, for all* x *in the neighbourhood*

(2) if $f(x, y) = 0$ *for* x, y *sufficiently close to 0, then* $y = g(x)$*.*

Proof. On the neighbourhood of $(0, 0)$, by Weierstraß preparation, we get

$$f(x, y) = (p_0(x) + y)r(x, y).$$

Now $r(x, y)$ is nowhere 0 on the neighbourhood, so $f(x, y) = 0$ if and only if $-p_0(x) = y$, so $g(x) = -p_0(x)$ will work. □

Corollary 4 ($k = 2$ in Weierstraß preparation)**.** *Let* $f : \overset{x}{\mathbb{C}} \times \overset{y}{\mathbb{C}} \to \mathbb{C}$ *and let* f *be analytic in a neighbourhood of* $(0, 0)$*. Suppose*

$$f(0, 0) = \frac{d}{dy} f(0, 0) = 0, \;\; \textit{but} \; \frac{d^2}{dy^2} f(0, 0) \neq 0.$$

Then in a neighbourhood of $(0, 0)$*,*

$$f(x, y) = (p_0(x) + p_1(x)y + y^2)r(x, y)$$

with p_i *analytic in neighbourhood and* $r(x, y)$ *nowhere 0 in the neighbourhood.*

Now we can prove Proposition 1:

Proof of Proposition 1. As $\exists \epsilon$ with $E(\rho+\epsilon, T(\rho)+\epsilon) < \infty$ and we have nonnegative coefficients, we can choose a neighbourhood $\mathcal{U}$ of $(\rho, T(\rho))$ such that E is analytic on $\mathcal{U}$.

Let
$$F(x,y) = y - E(x,y).$$
Then F is analytic on $\mathcal{U}$ and $F(x,T(x)) = T(x) - E(x,T(x)) = 0$ for $|x| < \rho$. By Pringsheim's Theorem, ρ is a singularity so the hypotheses of the implicit function theorem must be false at $(\rho, T(\rho))$; thus we must have $\frac{d}{dy}F(\rho,T(\rho)) = 0$.

$$\frac{d}{dy}F(x,y) = 1 - \frac{d}{dy}E(x,y)$$

We want to check that the hypotheses of Corollary 4 are satisfied:

$$\frac{d^2}{dy^2}F(x,y) = -\frac{d^2}{dy^2}E(x,y) < 0$$

for $x, y > 0$ (since we have nonnegative coefficients and at least one y^2 term), so in particular,

$$\frac{d^2}{dy^2}F(\rho,T(\rho)) < 0$$

thus
$$F(x,y) = \underbrace{(p_0(x) + p_1(x)y + y^2)}_{P(x,y)} r(x,y)$$

with p_i analytic not 0 at ρ and $r(x,y)$ analytic, nowhere 0 in a neighbourhood of $(\rho, T(\rho))$.

Let $D(x)$ be the discriminant of $P(x,y)$

$$D(x) = p_1(x)^2 - 4p_0(x)$$

Next we want to check $D(\rho) = 0$, $\frac{d}{dx}D(\rho) \neq 0$. To see these, just calculate:
$F(x,T(x)) = 0$, but $r(x,T(x)) \neq 0$ for x near ρ, so

$$p_0(\rho) + p_1(\rho)T(\rho) + T(\rho)^2 = 0. \tag{1}$$

Also

$$\begin{aligned} 0 &= \frac{d}{dy}F(\rho,T(\rho)) \\ &= \left(\frac{d}{dy}P(\rho,T(\rho))\right)\underbrace{r(\rho,T(\rho))}_{\neq 0} + \cancelto{0}{P(\rho,T(\rho))\frac{d}{dy}r(\rho,T(\rho))} \\ \Rightarrow 0 &= \frac{d}{dy}P(\rho,T(\rho)) \\ &= p_1(\rho) + 2T(\rho) \end{aligned}$$

and subbing into (1) gives

$$0 = p_0(\rho) - \frac{p_1^2(\rho)}{2} + \frac{p_1^2(\rho)}{4} = p_0(\rho) - \frac{p_1^2(\rho)}{4} = -\frac{D(\rho)}{4}$$

so $D(\rho) = 0$. Now

$$\begin{aligned}\frac{d}{dx}D(\rho) &= 2p_1(\rho)\frac{d}{dx}p_1(\rho) - 4\frac{d}{dx}p_0(\rho) \\ &= -4\left(T(\rho)\frac{d}{dx}p_1(\rho) + \frac{d}{dx}p_0(\rho)\right)\end{aligned}$$

$$\frac{d}{dx}F(\rho, T(\rho)) = -\frac{d}{dx}E(\rho, T(\rho)) < 0$$

and

$$\frac{d}{dx}F(\rho, T(\rho)) = \left(\frac{d}{dx}p_0(\rho) + T(\rho)\frac{d}{dx}p_1(\rho)\right) r(\rho, T(\rho)) + 0, \quad \text{since } P(\rho, T(\rho)) = 0,$$

So

$$\frac{d}{dx}D(\rho) = \frac{4\frac{d}{dx}E(\rho, T(\rho))}{r(\rho, T(\rho))} \neq 0.$$

Thus $D(\rho) = 0$, $\frac{d}{dx}D(\rho) \neq 0$.

Returning to the previous calculation we know

$$p_0(x) + p_1(x)T(x) + T(x)^2 = 0$$

for x near ρ, so

$$T(x) = -\frac{p_1(x)}{2} + \frac{1}{2}\sqrt{D(x)}.$$

Since $D(\rho) = 0$ we can expand $\sqrt{D(x)}$ around ρ to get

$$D(x) = \sum_{k \geq 1} d_k(\rho - x)^k$$

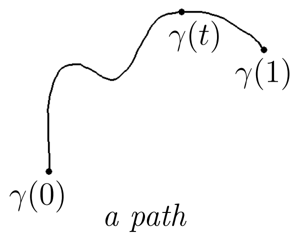
and since $\frac{d}{dx}D(\rho) \neq 0$ we know $d_1 \neq 0$. So

$$T(x) = \underbrace{-\frac{1}{2}p_1(x)}_{A(\rho - x)} + \underbrace{\left(\frac{1}{2}\sqrt{d_1}\sqrt{1 + \sum_{k \geq 1}\frac{d_{k+1}}{d_1}(\rho - x)^k}\right)}_{B(\rho - x)}\sqrt{\rho - x}$$

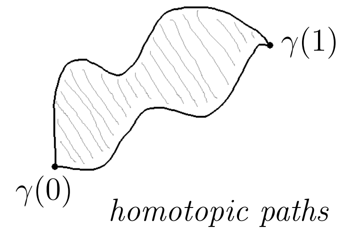
for x near ρ. □

2 Cauchy's Theorems

Definition. Let Ω be a connected open subset of $\mathbb{C}$. A *path* is a function $\gamma : [0,1] \to \Omega$.



a path



homotopic paths

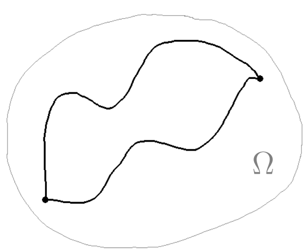
Definition. Two paths $\gamma_1, \gamma_2 : [0,1] \to \Omega$ with $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$ are *homotopic* (above right) if $\exists\, h(x,y)$ continuous with image in Ω such that

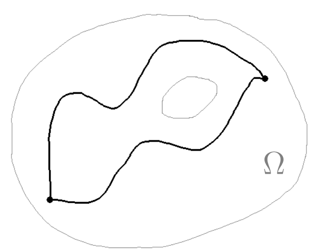
$$\begin{aligned} h(x,0) &= \gamma_1(x) \\ h(x,1) &= \gamma_2(x) \\ h(0,y) &= \gamma_1(0) \\ h(1,y) &= \gamma_1(1). \end{aligned}$$

Definition. A *closed path* has $\gamma(0) = \gamma(1)$.

Definition. A *simple path* is 1-1 as a function.

Note. Being homotopic depends on Ω.





Definition. Integrals along paths are defined as you'd expect:

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt.$$

Complex analysis is very rigid. Another important example of this is

Theorem 5. *If f is analytic on Ω and γ_1, γ_2 are homotopic in Ω then*

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Theorem 6 (Cauchy's residue theorem)**.** *Let $h(z)$ be meromorphic (i.e., holomorphic except possibly for finitely many poles) in Ω and let λ be a positively oriented simple closed path in Ω. Let $\mathcal{S}$ be the set of poles of h inside the region enclosed by λ. Then*

$$\frac{1}{2\pi i}\int_\lambda h(z)dz = \sum_{s\in\mathcal{S}} \mathrm{Res}_s h$$

where $\mathrm{Res}_s h$ *is the* $[(z-s)^{-1}]$ *in a Laurent expansion of h around s.*

Proof. (For just 1 pole at 0). So

$$h(z) = \sum_{n=-I}^{\infty} h_n z^n$$

then

$$\int_\lambda h(z)dz = \int_\lambda \sum_{\substack{n\geq -I\\ n\neq -1}} h_n z^n dz + h_{-1}\int_\lambda \frac{dz}{z}$$

and for $n \neq -1$,

$$\begin{aligned} h_n\int_\lambda z^n dz &= h_n\int_0^1 e^{2\pi i\,nt}\,2\pi i\,e^{2\pi it}dt, \quad \text{letting } \lambda(t)=e^{2\pi it} \\ &= 2\pi i\,h_n\int_0^1 e^{2\pi it(n+1)}dt \\ &= 0, \end{aligned}$$

but

$$\begin{aligned} \int_\lambda \frac{dz}{z} &= \int_0^1 e^{-2\pi i\,t}\,2\pi i\,e^{2\pi it}dt \\ &= 2\pi i\cdot 1. \end{aligned}$$

So $\int_\lambda h(z)dz = 2\pi i\,h_{-1}$. □

Theorem 7 (Cauchy's coefficient formula)**.** *Let $f(z)$ be analytic in a region Ω containing 0. Let λ be a positively oriented simple closed path in Ω. Then*

$$[z^n]f(z) = \frac{1}{2\pi i}\int_\lambda f(z)\frac{dz}{z^{n+1}}.$$

Proof. Write

$$f(z) = \sum_{\ell=0}^{\infty} f_\ell z^\ell$$

then

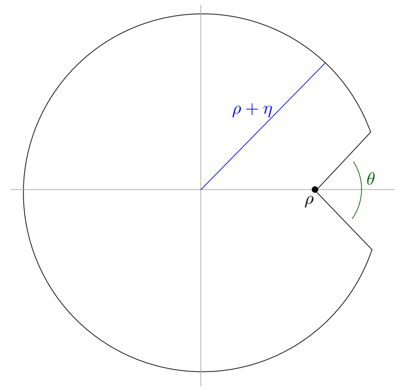
$$\frac{f(z)}{z^{n+1}} = \sum_{\ell=-n-1}^{\infty} f_{\ell+n+1} z^\ell$$

and so the residue is f_n, so the result is an application of Cauchy's residue theorem. □

3 Transfer Theorems

Now we can use this to get a nice transfer theorem.

Definition. A *delta neighbourhood of* ρ is a region as illustrated



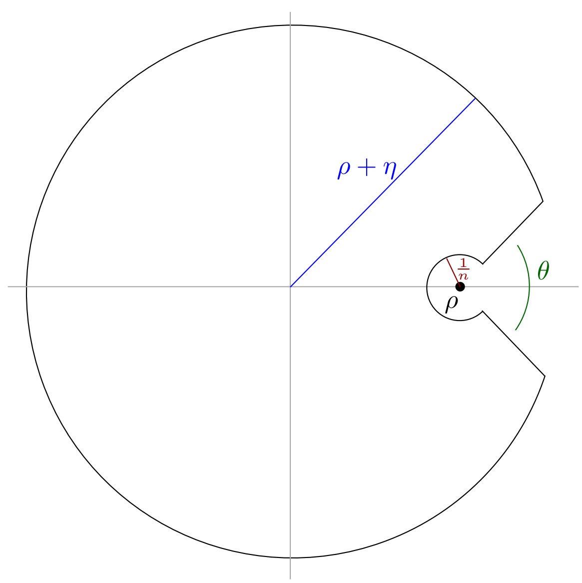
Note. Stirling's formula (with the constant) says for $\alpha \in \mathbb{R}\backslash\mathbb{Z}$

$$[x^n](\rho - x)^\alpha \sim \frac{\rho^\alpha}{\Gamma(-\alpha)}\rho^{-n}n^{-\alpha-1}.$$

Theorem 8 (Transfer theorem of Flajolet and Odlyzko)**.** *Let* $0 < \rho < \infty$ *and suppose* f *is analytic on* $\Delta - \rho$ *with* Δ *a delta neighbourhood of* ρ *and* $f(x) \sim K(\rho - x)^\alpha$ *as* $x \to \rho$ *in* Δ *with* $\alpha \in \mathbb{R}\backslash\mathbb{Z}$*, then*

$$\begin{aligned} [x^n]f(x) &\sim [x^n]K(\rho - x)^\alpha \\ &\sim \frac{K\rho^\alpha}{\Gamma(-\alpha)}\rho^{-n}n^{-\alpha-1} \end{aligned}$$

Sketch of proof. Use the following contour:



Write

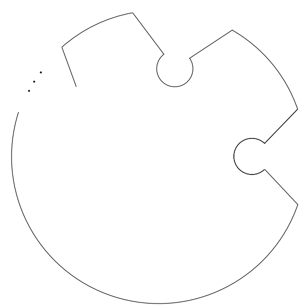
$$\gamma = \begin{cases} \gamma_1 = \{x : |x-\rho| = \frac{1}{n},\ |\arg(x-\rho)| \geq \theta\} & \text{inner circle} \\ \gamma_2 = \{x : \frac{1}{n} \leq |x-\rho|,\ |x| \leq \rho+\eta,\ \arg(x-\rho) = \theta\} & \text{straight piece} \\ \gamma_3 = \{x : |x| = \rho+\eta,\ |\arg(x-\rho)| \geq \theta\} & \text{outer circle} \\ \gamma_4 = \{x : \frac{1}{n} \leq |x-\rho|,\ |x| \leq \rho+\eta,\ \arg(x-\rho) = -\theta\} & \text{straight piece} \end{cases}$$

Wlog scale so $\rho = 1$. Now bound each piece:

- γ_1 bound by (length of path)(max of integrand)
- γ_2, γ_4 are tricky ones, like Γ-function integral
- γ_3 easy, as f bounded so only care about $\int \frac{1}{z^{n+1}}$

□

Note. If there are > 1 singularities on the circle of convergence, but only finitely many, we can give the same argument using the following contour (simply add more keyholes) to get the same result:



References

[1] Jason P. Bell, Stanley N. Burris, and Karen A. Yeats. Counting rooted trees: the universal law $t(n) \sim C\rho^{-n}n^{-3/2}$. *Electron. J. Combin.*, 13(1):Research Paper 63, 64 pp. (electronic), 2006.

[2] Philippe Flajolet and Robert Sedgewick. IV.2 VI.3. In *Analytic combinatorics*, pages xiv+810. Cambridge University Press, Cambridge, 2009.